Announcements

1) Quiz Thursday on 14.3 - 14.5

Example 1: Suppose

$$f(x,y) = x^3 + \sqrt{y}$$

$$x = x(s,t) = \ln(s+t)$$

$$y = y(s,t) = s^4 t^8.$$

We found:

$$\boxed{\begin{aligned} \frac{\partial g}{\partial s} &= \frac{3(\ln(s+t))^2}{s+t} + 2st^4 \\ \frac{\partial g}{\partial t} &= \frac{3(\ln(s+t))^2}{s+t} + 4s^2t^3 \end{aligned}}$$

But also:

$$\frac{\partial f}{\partial x} = 3x^2 \ , \ \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$$

$$\frac{\partial x}{\partial s} = \frac{1}{s+t} = \frac{\partial x}{\partial t}$$

$$\frac{\partial y}{\partial s} = 4s^3t^8 \ , \ \frac{\partial y}{\partial t} = 8s^4t^7$$

$$\frac{\partial f}{\partial x}(s,t) = 3(\ln(s+t))^2$$

$$\frac{\partial f}{\partial y}(s,t) = \frac{1}{2\sqrt{s^4t^8}} = \frac{1}{2s^2t^4}$$

Claim: If

$$g(s,t) = f(x(s,t), y(s,t)),$$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}(s,t)\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}(s,t)\frac{\partial y}{\partial s}$$

$$= (3\ln(s+t))^2\left(\frac{1}{s+t}\right) + \frac{1}{2s^2t^4}\left(4s^3t\right)$$

$$= \frac{3(\ln(s+t))^2}{s+t} + 2st^4$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}(s,t)\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(s,t)\frac{\partial y}{\partial t}$$

$$= 3(\ln(s+t))^2\left(\frac{1}{s+t}\right) +$$

$$\frac{1}{2s^2t^4}(8s^4t^7)$$

$$= \frac{3(\ln(s+t))^2}{s+t} + 4s^2t^3$$

Chain Rule (Adolescent version)

Given $z = f(x,y)$ and

$x = x(s,t)$, $y = y(s,t)$. Then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

(understood that you should compose $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ with $x(s,t), y(s,t)$)

Implicit Differentiation

Suppose F is a function of 3 variables, consider

$$F(x,y,z)=C$$

for some constant C. Then another version of the implicit function theorem says we can solve for z in terms of x and y provided F is "sufficiently nice" in a small solid sphere about (x,y,z).

In this case,

$z = z(x,y)$, and so

by the chain rule,

if $g(x,y) = F(x,y,z(x,y))$,

$$0 = \frac{\partial g}{\partial x} = \frac{\partial F}{\partial x}\overbrace{\frac{\partial x}{\partial x}}^{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}, \text{ so}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\left(-\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)}}$$

Then similarly,

$$\boxed{\frac{\partial z}{\partial y} = \frac{\left(-\frac{\partial f}{\partial y}\right)}{\left(\frac{\partial f}{\partial z}\right)}}$$

Tangent Planes and the Gradient

Section 14.4

In one dimension, the derivative is the slope of the tangent line.

What do the partial derivatives represent?

Definition: (gradient)

Given a function $z = f(x,y)$ from some domain D in $\mathbb{R}^2$, the **gradient** of f at a point (a,b) in D is denoted by $\nabla f(a,b)$, and is equal to

$$\boxed{\nabla f(a,b) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle}$$

Example 2:

$f(x,y) = 1 - x^2 - y^2$

Find $\nabla f(0,2)$.

$\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial y} = -2y$,

so

$\frac{\partial f}{\partial x}(0,2) = 0$, $\frac{\partial f}{\partial y}(0,2) = -4$,

and

$$\boxed{\nabla f(0,2) = \langle 0, -4 \rangle}$$

The Gradient and Orthogonality

Given a function $z = f(x,y)$;

choosing a constant c,

set $c = f(x,y)$ (gives a level curve of f). We

say that a vector is orthogonal to the level curve at a point (a,b) on the curve if the vector is orthogonal to the direction vector of the tangent line to the curve at (a,b)

Example 3: $f(x,y) = 1 - x^2 - y^2$

Consider the level curve

$$0 = 1 - x^2 - y^2 \quad (c = 0)$$

Find the tangent vector to the curve at the point $(1/2, \sqrt{3}/2)$, Show it is orthogonal to $\nabla f(1/2, \sqrt{3}/2)$.

By the chain rule,

$$\frac{dy}{dx} = \frac{\left(-\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

$$= \frac{2x}{-2y} = -\frac{x}{y},$$

take direction vector

$\langle y, -x \rangle$. The gradient is

$\nabla f = \langle -2x, -2y \rangle$ Evaluate

both at $(1/2, \sqrt{3}/2)$.

Tangent vector: $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

Gradient : $\langle -1, -\sqrt{3} \rangle$

$$\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle \cdot \langle -1, -\sqrt{3} \rangle$$

$$= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 0 \checkmark$$

In General

If $z = f(x,y)$ and c is a constant, then for the curve $c = f(x,y)$,

$$\frac{dy}{dx} = \frac{\left(-\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)},$$

so a tangent vector is given by

$$\left\langle \frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x} \right\rangle$$

Then

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle,$$ and

so at any point (a,b) on the level curve,

$$\nabla f(a,b) \cdot \left\langle \frac{\partial f}{\partial y}(a,b), -\frac{\partial f}{\partial x}(a,b) \right\rangle$$

$$= \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle \cdot \left\langle \frac{\partial f}{\partial y}(a,b), -\frac{\partial f}{\partial x}(a,b) \right\rangle$$

$$= \boxed{0}$$

This shows $\nabla f(a,b)$ is orthogonal to the level curve at (a,b).

To get the equation of the tangent line:

$$\boxed{\nabla f(a,b) \cdot \langle x-a, y-b \rangle = 0}$$

Example 4: (tangent planes)

$$x+2y+z^2 = F(x,y,z)$$

@ $(0,1,2)$

Choose $c=6$. Then

$6 = x+2y+z^2$ is a

level surface of

$F(x,y,z)$. In small areas

of the graph, F looks

2-dimensional.

Instead of a tangent line, which is one-dimensional, the level surface should have tangent planes.

This will be the plane generated by two tangent vectors, as follows: intersect the graph of $6 = x + 2y + z^2$ with the plane $x = 0$ to get a curve $6 = 2y + z^2$.

Solving for y,

$$y = \frac{6-z^2}{2}, \quad \text{so}$$

$$\frac{dy}{dz} = -z, \quad \text{at } (0,1,2)$$

this is equal to -2, so

a tangent vector would be

$\langle -2, 1 \rangle$. Note that

by the chain rule,

$$\frac{\partial y}{\partial z} = \frac{\left(-\frac{\partial F}{\partial z}\right)}{(\partial F/\partial y)} = \frac{-2z}{2} = -z!$$

So then

$$\frac{\partial y}{\partial z}(0,1,2) = \frac{dy}{dz}(0,1,2)$$

if we fix $x=0$.

Cheat for the other vector using this observation:

When $y=1$

$$\frac{dx}{dz} = \frac{\left(-\frac{\partial F}{\partial z}\right)}{\left(\partial F/\partial x\right)} = \frac{-2z}{1} = -2z$$

So at $(0,1,2)$, $\frac{dx}{dz} = -4$

We then have two tangent vectors:

$$\langle \underset{y}{-2}, \underset{z}{1} \rangle \quad \text{and}$$

$$\langle \underset{x}{-4}, \underset{z}{1} \rangle$$

Include these vectors into $\mathbb{R}^3$ by putting a zero in the missing coordinate:

$$\langle \underset{y}{-2}, \underset{z}{1} \rangle \longmapsto \langle 0, -2, 1 \rangle$$

$$\langle \underset{x}{-4}, \underset{z}{1} \rangle \longmapsto \langle -4, 0, 1 \rangle$$

We have a plane generated by $v = \langle 0, -2, 1 \rangle$ and $w = \langle -4, 0, 1 \rangle$. We want the parallel plane containing $(0, 1, 2)$. We could cross v and w to get a normal vector or observe

$$F(x,y,z) = x + 2y + z^2$$

$$\nabla F = \langle 1, 2, 2z \rangle$$

$$\nabla F(0,1,2) = \langle 1, 2, 4 \rangle.$$

We see that

$$\nabla F(0,1,2) \cdot \langle 0,-2,1 \rangle$$

$$= \langle 1,2,4 \rangle \cdot \langle 0,-2,1 \rangle = 0$$

and

$$\nabla F(0,1,2) \cdot \langle -4,0,1 \rangle$$

$$= \langle 1,2,4 \rangle \cdot \langle -4,0,1 \rangle = 0$$

Equation for tangent plane:

$$\nabla F(0,1,2) \cdot \langle x, y-1, z-2 \rangle = 0$$

$$= \boxed{\langle 1,2,4 \rangle \cdot \langle x, y-1, z-2 \rangle = 0}$$

The Tangent Plane

Given a function $w = F(x,y,z)$, consider the level surface

$$c = F(x,y,z)$$

for some constant c.

Then the tangent plane at a point (x_0, y_0, z_0) on the surface is

$$\boxed{\nabla F(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0}$$

Example 5: If

$$f(x,y,z) = \sin(xy^2z) + x - y$$

and $c = 1$, find the tangent plane to $1 = \sin(xy^2z) + x - y$ at $(1, 1, \pi/2)$.

$$\nabla f = \langle y^2z\cos(xy^2z) + 1,$$

$$2xyz\cos(xy^2z) - 1,$$

$$xy^2\cos(xy^2z)\rangle$$

$$\nabla f(1,1,\pi/2) = \langle 1, -1, 0\rangle$$

Tangent plane:

$$\boxed{\langle 1, -1, 0 \rangle \cdot \langle x-1, y-1, z-\frac{\pi}{2} \rangle = 0}$$

For Functions

For $z = f(x,y)$, consider the function $G(x,y,z) = f(x,y) - z$.

$z = f(x,y)$ corresponds to $G(x,y,z) = 0$, so the tangent plane to the graph of f at $(x_0, y_0, f(x_0,y_0))$ is

$$\nabla G(x_0, y_0, f(x_0,y_0)) \cdot \langle x - x_0, y - y_0, z - f(x_0,y_0) \rangle$$

$$= 0$$

$$\boxed{0 = \left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1 \right\rangle \cdot \langle x - x_0, y - y_0, z - f(x_0,y_0) \rangle}$$

Example 6: Let

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Find the equation of the tangent plane to the graph of f at the point $(3,4)$.

$$\nabla f = \left\langle \frac{y\sqrt{x^2+y^2} - \frac{x^2y}{\sqrt{x^2+y^2}}}{x^2+y^2}, \frac{x\sqrt{x^2+y^2} - \frac{y^2x}{\sqrt{x^2+y^2}}}{x^2+y^2} \right\rangle$$

$$= \frac{1}{(x^2+y^2)^{3/2}} \left\langle y^3, x^3 \right\rangle$$

$$\nabla f(3,4) = \frac{1}{25^{3/2}} \langle 64, 27 \rangle$$

$$= \frac{1}{125} \langle 64, 27 \rangle$$

Tangent plane:

$$\frac{1}{125}\langle 64, 27, -1 \rangle \cdot \langle x-3, y-4, z-f(3,4) \rangle = 0$$

$$f(3,4) = \frac{12}{5}, \text{ so}$$

$$\boxed{\frac{1}{125}\langle 64, 27, -1 \rangle \cdot \langle x-3, y-4, z-\frac{12}{5} \rangle = 0}$$